

# Missing solution in a relativistic Killingbeck potential

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**Abstract.** Missing bound-state solutions for fermions in the background of a Killingbeck radial potential including an external magnetic and Aharonov–Bohm (AB) flux fields are examined. The correct quadratic form of the Dirac equation with vector and scalar couplings under the spin and pseudo-spin symmetries is showed and also we point out a misleading treatment in the literature regarding to bound-state solutions for this problem.

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## 1 Introduction

The study of low-dimensional fermion systems has long been recognized as important in understanding various phenomena in different areas of physics, such as Particle physics, Condensed matter physics, among others. One of the main reasons is than it provides a simplified version of the three-dimensional world. However, these systems are more than a prototype of the higher-dimensional systems and are being scenario of new and exotics phenomena that have generated a great interest, both theoretical and applied physics.

On the theoretical side, the fermion number (charge) fractionalization in relativistic quantum field theory is a remarkable phenomenon and that was shown to occur in one dimensional systems [1] (as polyacetylene [2–4]) when fermions interact with background fields with a topologically nontrivial soliton profile. In this context, isolated zero modes (isolated solutions) of the fermion–soliton system can have fractional fermion numbers of  $\pm 1/2$ . Similarly, the existence of these isolated solutions in bidimensional condensed matter systems are also responsible to induce a fractional charge. For example, in [5] the authors shown that when lattice distortions with vortex profile are incorporated in a graphene lattice, there are zero modes excitations in the single-particle energy spectrum. From the existence of isolated zero modes and from the sublattice symmetry, they show that the fermion quantum charge is fractionalized, which even persists if [5] is extended to a chiral gauge theory for graphene [6–8].

In this paper we consider some interactions used in [5–8]. Particularly, we study the dynamics of fermions in  $2+1$  dimensions under a influence of mixture of scalar  $S(r)$ , vector  $V(r)$  and minimal  $A(r)$  interactions. The classi-

fication of the potentials are based on the behavior under a Lorentz transformation:  $S(r)$  for the Lorentz scalar, and  $V(r)$  and  $A(r)$  for the time and space components of a two-vector potential, respectively. Each of these interactions has important applications and the study of its effects on the dynamics of fermions are of great interest in the scientific community [9]. The time and space components of a two-vector potential is useful for studying the dynamics of a spin-1/2 charged particle in an electric and magnetic fields, respectively. On the other hand, the scalar potential can be interpreted as a position-dependent mass.

The case in which the couplings are composed by a vector  $V(r)$  and a scalar  $S(r)$  potentials, with  $S(r) = V(r)$  [or  $S(r) = -V(r)$ ], are usually pointed out as necessary condition for occurrence of exact spin [or pseudo-spin] symmetry. It is known that the spin and pseudo-spin symmetries are  $SU(2)$  symmetries of a Dirac Hamiltonian with vector and scalar potentials. The pseudo-spin symmetry was introduced in nuclear physics many years ago [10, 11] to account for the degeneracies of orbital in single-particle spectra. Also, it is known that the spin symmetry occurs in the spectrum of a meson with one heavy quark [12] and anti-nucleon bound in a nucleus [13], and the pseudo-spin symmetry occurs in the spectrum of nuclei [14].

In a recent article in this journal [15], Eshghi and collaborators investigated the Dirac equation in  $2+1$  dimensions with a Killingbeck radial potential including an external magnetic and Aharonov–Bohm (AB) flux fields. They mapped the Dirac equation into Sturm–Liouville problem of a Schrödinger-like equation and obtained a set of bound-state solutions by recurring to the properties of the biconfluent Heun equation. Nevertheless, an isolated solution from the Sturm–Liouville scheme was not taken

into account. The purpose of this work is to report on this missing bound-state solution and additionally we point out a misleading treatment in Ref. [15]. Additionally, we shed some light on a misconception recently propagated in the literature with respect to Aharonov–Bohm (AB) potential.

## 2 Dirac equation in 2 + 1 dimensions

The Dirac equation in 2 + 1 dimensions in polar coordinates is given by ( $\hbar = c = 1$ )

$$\{\beta \boldsymbol{\gamma} \cdot \boldsymbol{\pi} + \beta [M + S(r)]\} \psi(\mathbf{r}) = [E - V(r)] \psi(\mathbf{r}), \quad (1)$$

where  $\boldsymbol{\pi} = (\pi_r, \pi_\varphi) = (-i\partial_r, -i\partial_\varphi/r - eA_\varphi)$ ,  $\mathbf{r} = (r, \varphi)$  and  $\psi$  is a two-component spinor. Here  $E$  is the energy of the fermion,  $S(r)$  is a scalar potential,  $V(r)$  is the time-like vector potential and  $A_\varphi$  is the space-like vector potential. The  $\boldsymbol{\gamma}$  matrices in Eq. (1) are given in terms of the Pauli matrices as [16]

$$\beta \boldsymbol{\gamma}^r = \sigma_1 \cos \varphi + s \sigma_2 \sin \varphi = \begin{pmatrix} 0 & e^{-is\varphi} \\ e^{+is\varphi} & 0 \end{pmatrix}, \quad (2)$$

$$\beta \boldsymbol{\gamma}^\varphi = -\sigma_1 \sin \varphi + s \sigma_2 \cos \varphi = \begin{pmatrix} 0 & -ise^{-is\varphi} \\ ise^{+is\varphi} & 0 \end{pmatrix}, \quad (3)$$

$$\beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4)$$

where  $s$  is twice the spin value, with  $s = +1$  for spin “up” and  $s = -1$  for spin “down”. It is worthwhile to mention that the representations (2), (3) and (4) are most suitable for 2 + 1 dimensions. Equation (1) can be written more explicitly as

$$e^{-is\varphi} [\pi_r - is\pi_\varphi] \psi_2 = [E - M - \Sigma(r)] \psi_1, \quad (5)$$

$$e^{+is\varphi} [\pi_r + is\pi_\varphi] \psi_1 = [E + M - \Delta(r)] \psi_2, \quad (6)$$

where  $\Sigma(r) = V(r) + S(r)$  and  $\Delta(r) = V(r) - S(r)$ .

If one adopts the following decomposition

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{r}} \begin{pmatrix} \sum_m f_m(r) e^{im\varphi} \\ i \sum_m g_m(r) e^{i(m+s)\varphi} \end{pmatrix}, \quad (7)$$

with  $m+1/2 = \pm 1/2, \pm 3/2, \dots$ , with  $m \in \mathbb{Z}$ , and inserting this into Eqs. (5) and (6), we obtain

$$\begin{aligned} \left[ \frac{d}{dr} + \frac{s(m+s) - \frac{1}{2}}{r} - esA_\varphi \right] g_m &= [E - M - \Sigma] f_m, \\ \left[ -\frac{d}{dr} + \frac{sm + \frac{1}{2}}{r} - esA_\varphi \right] f_m &= [E + M - \Delta] g_m. \end{aligned} \quad (8)$$

Note that it is impossible to obtain the equation (9) [or equation (8)] from equation (8) [or equation (9)] by mean of a charge conjugation or discrete chiral transformation, as already was uncovered in Ref. [17–19] for 1 + 1 dimensions, and in Ref. [20] for 3 + 1 dimensions.

For  $\Sigma(r) = 0$  [or  $\Delta(r) = 0$ ] with  $E \neq M$  [or  $E \neq -M$ ], the searching for solutions can be formulated as a Sturm–Liouville problem for the component  $g(r)$  [or  $f(r)$ ] of the Dirac spinor, as done in Ref. [15] for bound states. On the other hand, the solutions for  $\Delta(r) = 0$  with  $E = -M$  and  $\Sigma(r) = 0$  with  $E = M$ , excluded from the Sturm–Liouville problem, can be obtained directly from the first-order equations (8) and (9), such solutions are called isolated solutions [18, 21–27].

## 3 Isolated solutions for the Dirac equation in 2 + 1 dimensions

For  $\Sigma(r) = 0$  with  $E = M$ , the first-order equations (8) and (9) reduce to

$$\left[ \frac{d}{dr} + \frac{s(m+s) - \frac{1}{2}}{r} - seA_\varphi \right] g_m = 0, \quad (10)$$

$$\left[ -\frac{d}{dr} + \frac{sm + \frac{1}{2}}{r} - seA_\varphi \right] f_m = 2(M - V) g_m, \quad (11)$$

whose general solution is

$$g_m(r) = a_+ r^{-s(m+s) + \frac{1}{2}} e^{sev(r)}, \quad (12)$$

$$f_m(r) = [b_+ - a_+ I(r)] r^{sm + \frac{1}{2}} e^{-sev(r)}, \quad (13)$$

where  $a_+$  and  $b_+$  are normalization constants, and

$$I(r) = \int dr [2M - 2V(r)] r^{-(2sm+1)} e^{2sev(r)}, \quad (14)$$

$$v(r) = \int^r A_\varphi(x) dx. \quad (15)$$

Note that this sort of isolated solution cannot describe scattering states and is subject to the normalization condition

$$\int_0^\infty (|f_m(r)|^2 + |g_m(r)|^2) dr = 1. \quad (16)$$

Because  $f_m(r)$  and  $g_m(r)$  are normalize functions, the possible isolated solution presupposes  $A_\varphi \neq 0$ . This fact clearly shows that the normalization of the isolated solution is decided by the behavior of  $v(r)$ , i.e. the presence of  $A_\varphi$  is an essential ingredient for the normalization of the isolated solution. Observing (12) and (13), one can conclude that it is impossible to have both nonzero components simultaneously as physically acceptable solutions.

For  $\Delta(r) = 0$  with  $E = -M$ , the first-order equations (8) and (9) reduce to

$$\left[ \frac{d}{dr} + \frac{s(m+s) - \frac{1}{2}}{r} - esA_\varphi \right] g_m = -2[M + V] f_m, \quad (17)$$

$$\left[ -\frac{d}{dr} + \frac{sm + \frac{1}{2}}{r} - esA_\varphi \right] f_m = 0, \quad (18)$$

whose general solution is

$$f_m(r) = a_- r^{sm+\frac{1}{2}} e^{-sev(r)}, \quad (19)$$

$$g_m(r) = [b_- - a_- H(r)] r^{-s(m+s)+\frac{1}{2}} e^{sev(r)}, \quad (20)$$

where  $a_-$  and  $b_-$  are normalization constants, and

$$H(r) = \int dr [2M - 2V(r)] r^{2sm+1} e^{-2sev(r)}. \quad (21)$$

The same conclusions obtained from the case  $\Sigma(r) = 0$  with  $E = M$  are valid, i.e. the presence of  $A_\varphi$  is an essential ingredient for the normalization of the isolated solution and it is impossible to have both nonzero components simultaneously.

Having set up the Dirac equation in 2 + 1 dimensions, we are now in a position to use the machinery developed above in order to find isolated solutions with some specific forms for the external interactions. Assuming the external interactions as in Ref. [15],

$$\mathbf{A} = \left(0, \frac{B_0 r}{2} + \frac{\Phi_{AB}}{2\pi r}, 0\right), \quad (22)$$

$$V(r) = ar^2 + br - \frac{c}{r}, \quad (23)$$

where  $B_0$  is the magnetic field magnitude,  $\Phi_{AB}$  is the flux parameter,  $a$ ,  $b$  and  $c$  are constants.

In this case, substituting (22) in (15) one finds

$$v(r) = \frac{B_0 r^2}{4} + \frac{\Phi_{AB}}{2\pi} \ln r. \quad (24)$$

Now, using (24) we are now in a position to find the isolated solutions for  $\Sigma(r) = 0$  with  $E = M$  and  $\Delta(r) = 0$  with  $E = -M$ .

### 3.1 Isolated solution for $\Sigma(r) = 0$ with $E = M$

In this case, the solutions (12) and (13) become

$$g_m(r) = a_+ r^{s(\lambda-m)-\frac{1}{2}} e^{s\delta r^2}, \quad (25)$$

$$f_m(r) = [b_+ - a_+ I(r)] r^{-s(\lambda-m)+\frac{1}{2}} e^{-s\delta r^2}, \quad (26)$$

where  $\lambda = \frac{e\Phi_{AB}}{2\pi}$  and  $\delta = \frac{eB_0}{4}$ . In this case, for  $\lambda > 0$ ,  $\delta > 0$  and  $s = 1$ , a normalizable solution is possible only for  $a_+ = 0$ . Therefore,

$$\begin{pmatrix} f_m \\ g_m \end{pmatrix} = b_+ r^{m-\lambda+\frac{1}{2}} e^{-\delta r^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (27)$$

independently of  $a$ ,  $b$ ,  $c$  and  $M$ . Note that (27) is square-integrable at the origin and satisfies  $g_m(0) = 0$  when  $m - \lambda + \frac{1}{2} > 0$ .

In the case, for  $\lambda > 0$ ,  $\delta > 0$  and  $s = -1$ , a normalizable solution requires  $b_+ = 0$ , and a good behavior of  $I(r)$ . For the Killingbeck potential (23),  $I(r)$  can be expressed in terms of the incomplete gamma function [28]

$$\gamma(\alpha, r) = \int_0^r dt e^{-t} t^{\alpha-1}, \quad \text{Re } \alpha > 0. \quad (28)$$

Because  $\gamma(\alpha, r)$  tends to  $\Gamma(\alpha)$  as  $r \rightarrow \infty$ ,  $f_m$  is not, in general, a square-integrable function. So, a normalizable solution occurs when  $M = a = b = c = 0$  ( $I(r) = 0$ ). Therefore,

$$\begin{pmatrix} f_m \\ g_m \end{pmatrix} = a_+ r^{m-\lambda-\frac{1}{2}} e^{-\delta r^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (29)$$

Note that in this case, the solution (29) is square-integrable at the origin and satisfies  $f_m(0) = 0$  when  $m - \lambda - \frac{1}{2} > 0$ .

### 3.2 Isolated solution for $\Delta(r) = 0$ with $E = -M$

In this case, the solutions (19) and (20) become

$$f_m(r) = a_- r^{-s(\lambda-m)+\frac{1}{2}} e^{-s\delta r^2}, \quad (30)$$

$$g_m(r) = [b_- - a_- H(r)] r^{s(\lambda-m)-\frac{1}{2}} e^{s\delta r^2}. \quad (31)$$

Following the same procedure of the previous case, for  $s = -1$  a normalizable solution occurs when  $a_- = 0$ . The isolated solution is given by

$$\begin{pmatrix} f_m \\ g_m \end{pmatrix} = b_- r^{m-\lambda-\frac{1}{2}} e^{-\delta r^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (32)$$

with  $m - \lambda - \frac{1}{2} > 0$ .

For  $s = 1$  a normalizable solution is possible only for  $b_- = M = a = b = c = 0$ . In this case, the normalizable solution is given by

$$\begin{pmatrix} f_m \\ g_m \end{pmatrix} = a_- r^{m-\lambda+\frac{1}{2}} e^{-\delta r^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (33)$$

with  $m - \lambda + \frac{1}{2} > 0$ .

## 4 Quadratic form of the Dirac equation in 2 + 1 dimensions

Now, we investigate the dynamics for  $E \neq \pm M$ . For this, we choose to work with Eq. (1) in its quadratic form. After application of the operator

$$\beta [(M + S(r)) + \beta (E - V(r)) + \boldsymbol{\gamma} \cdot \boldsymbol{\pi}], \quad (34)$$

we get [29]

$$\begin{aligned} & \left\{ \mathbf{p}^2 - 2e(\mathbf{A} \cdot \mathbf{p}) + e^2(\mathbf{A})^2 \right\} \psi(\mathbf{r}) \\ & + \left\{ [M + S(r)]^2 - [E - V(r)]^2 - e\boldsymbol{\sigma} \cdot \mathbf{B} \right\} \psi(\mathbf{r}) \\ & - \left( \frac{\partial S(r)}{\partial r} \sigma_2 + i \frac{\partial V(r)}{\partial r} \sigma_1 \right) \psi(\mathbf{r}) = 0. \end{aligned} \quad (35)$$

In this stage, it is worthwhile to mention that the Eq. (35) is the correct quadratic form of the Dirac equation with minimal, vector and scalar couplings, because the Pauli term is considered.

Now, we focus attention on a misconception diffused in the literature. The vector potential in (22) furnishes one magnetic field perpendicular to the plane  $(r, \varphi)$ , given by [30–32]

$$\mathbf{B} = \nabla \times \mathbf{A} = \left( B_0 + \frac{\Phi_{AB}\delta(r)}{2\pi r} \right) \hat{z}, \quad (36)$$

and not simply  $\mathbf{B} = B_0 \hat{z}$  as considered in the Refs. [15, 33–41]. Here, we can interpret the first term in (36) as an constant external magnetic field and the second term as the magnetic field produce by a solenoid. If the solenoid is extremely long, the field inside is uniform, and the field outside is zero. However, in a general dynamics, the particle is allowed to access the  $r = 0$  region. In this region, the magnetic field is non-null. If the radius of the solenoid is  $r_0 \approx 0$ , then the relevant magnetic field is  $\mathbf{B} \sim \delta(r)$ . Therefore, on the study of the dynamics of a particle with spin, such term cannot be neglected in the equation of motion [30], because has important implications on the physical quantities of interest, such as energy eigenvalues, scattering matrix and phase shift (see Ref. [42] for more details). This situation has not been accomplished in Refs. [15, 39].

#### 4.1 Exact spin symmetry limit: $S(r) = V(r)$

By using the condition  $S(r) = V(r)$  ( $\Delta(r) = 0$ ) in (35), we obtain a second order differential equation for  $\psi_1$ . In this case, the upper component of the Dirac spinor can be considered as

$$\psi_1 = \sum_m \frac{f_m(r)}{\sqrt{r}} e^{im\varphi}. \quad (37)$$

So, substituting (22), (23), (36) and (37) in (35), the equation for  $f_m(r)$  becomes

$$H f_m(r) = k^2 f_m(r), \quad (38)$$

with

$$H = H_0 - \frac{es\Phi_{AB}\delta(r)}{2\pi r}, \quad (39)$$

$$H_0 = -\frac{d^2}{dr^2} + \eta r^2 + \rho r + \frac{\nu}{r^2} - \frac{\mu}{r}, \quad (40)$$

where

$$\eta = 2(E + M)a + \frac{e^2 B_0^2}{4}, \quad (41)$$

$$\rho = 2(E + M)b, \quad (42)$$

$$\nu = \left( m - \frac{e\Phi_{AB}}{2\pi} \right)^2 - \frac{1}{4}, \quad (43)$$

$$\mu = 2(E + M)c, \quad (44)$$

$$k^2 = E^2 - M^2 + e(m + s)B_0 - \frac{e^2 B_0 \Phi_{AB}}{2\pi}. \quad (45)$$

Note that the equation (38) depends on the spin projection  $s$  and it is different from that given in Ref. [15]. It is worthwhile to mention that (38) is the correct equation

of motion for the upper component of the Dirac spinor under the exact spin symmetry limit.

The solution considering only the term  $H_0$  (40), with  $\eta$  necessarily real and positive, is the solution of the Schrödinger equation for the three-dimensional harmonic oscillator plus a Cornell potential [20], which can be obtained by recurring to the properties of the biconfluent Heun equation. This potential was considered in Refs. [43, 44], but the authors misunderstood the full meaning of the potential and made a few erroneous calculations.

The presence of a  $\delta(r)$  interaction in the radial Hamiltonian (38) makes the problem more complicated to resolve. For this case, the most adequate procedure to address this problem is by means of the self-adjoint extension approach [45], but unhappily the self-adjoint extension for a biconfluent Heun equation is unknown.

#### 4.2 Exact pseudo-spin symmetry limit: $S(r) = -V(r)$

By using the condition  $S(r) = -V(r)$  ( $\Sigma(r) = 0$ ) in (35), we obtain a second order differential equation for  $\psi_2$ . In this case, the lower component of the Dirac spinor can be considered as

$$\psi_2 = i \sum_m \frac{g_m(r)}{\sqrt{r}} e^{i(m+s)\varphi}. \quad (46)$$

So, substituting (22), (23), (36) and (46) in (35), the equation for  $g_m(r)$  becomes

$$\tilde{H} g_m(r) = \tilde{k}^2 g_m(r), \quad (47)$$

with

$$\tilde{H} = \tilde{H}_0 - \frac{es\Phi_{AB}\delta(r)}{2\pi r}, \quad (48)$$

$$\tilde{H}_0 = -\frac{d^2}{dr^2} + \tilde{\eta} r^2 + \tilde{\rho} r + \frac{\tilde{\nu}}{r^2} - \frac{\tilde{\mu}}{r}, \quad (49)$$

where

$$\tilde{\eta} = 2(E - M)a + \frac{e^2 B_0^2}{4}, \quad (50)$$

$$\tilde{\rho} = 2(E - M)b, \quad (51)$$

$$\tilde{\nu} = \left( m + s - \frac{e\Phi_{AB}}{2\pi} \right)^2 - \frac{1}{4}, \quad (52)$$

$$\tilde{\mu} = 2(E - M)c, \quad (53)$$

$$\tilde{k}^2 = E^2 - M^2 + e(m + 2s)B_0 - \frac{e^2 B_0 \Phi_{AB}}{2\pi}. \quad (54)$$

The equation (47) is the correct equation of motion for the lower component of the Dirac spinor under the exact pseudo-spin symmetry limit and again it is different from that given in Ref. [15]. Analogously to the previous case, (47) depends on the spin projection  $s$ .

## 5 Final remarks

In this paper, we reinvestigated the issue of the Dirac equation in  $2 + 1$  dimensions with a Killingbeck radial potential including an external magnetic and Aharonov–Bohm (AB) flux fields. Using an adequate representation for the Dirac matrices, we solved the first order Dirac equation and found solutions for  $\Sigma(r) = 0$  with  $E = M$  and  $\Delta(r) = 0$  with  $E = -M$ , which are called isolated solutions because they are excluded from the Sturm–Liouville scheme. We showed that these solutions depend on the spin projection parameter  $s$  and that the presence of the vector potential is indispensable for a normalizable isolated solution. Also, we pointed out a misleading treatment recently propagated in the literature with respect to Aharonov–Bohm (AB) potential. Finally, we also showed the correct quadratic form of the Dirac equation in  $2 + 1$  dimensions taking into account the spin and pseudo-spin symmetries, which includes a  $\delta(r)$  function as a consequence of the Pauli term. It is known that to properly study the dynamics of the particle in this case, the most adequate procedure is the self-adjoint extension approach [45], but unhappily the self-adjoint extension for a biconfluent Heun equation is unknown. This last problem is open.

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